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COMPACT CONFORMALLY FLAT RIEMANNIAN MANIFOLDS

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1. Introduction

Sufficient conditions for a conformally flat Riemannian manifold to be a space of constant curvature were given by M. Tani [15] and S. I. Goldberg [5], etc. In this paper we study compact conformally flat Riemannian manifolds with finite fundamental group, and obtain the following theorems. Throughout this paper manifolds are assumed to be connected, of class C^∞, and of dimension $m \geq 3$.

Theorem A. *Let (M^m, g) be a compact conformally flat Riemannian manifold with finite fundamental group. If the scalar curvature S of (M^m, g) is constant, then S is positive and (M^m, g) is of constant curvature.*

If a complete Riemannian manifold (M^m, g) is of positive Ricci curvature $(\geq \varepsilon > 0)$, M^m is compact and has finite fundamental group (S. B. Myers [10]). Hence, as a natural consequence of Theorem A, we have M. Tani's theorem:

Corollary [15]. *Let (M^m, g) be a compact orientable conformally flat Riemannian manifold. Then (M^m, g) is of constant curvature if its Ricci curvature is positive and scalar curvature is constant.*

By $S^m(K)$ we denote a Euclidean m-sphere of constant curvature K.

Theorem B. *Let (M^m, g) be a compact conformally flat Riemannian manifold with finite fundamental group and constant scalar curvature S. If (M^m, g) admits a nonisometric conformal transformation, then (M^m, g) is isometric to $S^m(K)$ where $K = S/[m(m-1)]$.*

It is well known that there are no harmonic p-forms for $0 < p < m$ on a compact orientable conformally flat Riemannian manifold (M^m, g) of positive Ricci curvature; thus (M^m, g) is a rational homology sphere ([3], [9], see also [18, Theorem 4.1]). A more precise statement is as follows.

If a complete conformally flat Riemannian manifold (M^m, g) is of positive Ricci curvature $(\geq \varepsilon > 0)$, then the universal covering Riemannian manifold $({}^*M^m, {}^*g)$ of (M^m, g) is conformorphic to $S^m(1)$.

With respect to this we have

Theorem C. *The following three are equivalent:*

(i) *On a compact conformally flat Riemannian manifold (M^m, g) with finite*

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fundamental group, we have a conformally related Riemannian metric g^* *of constant curvature.*

(ii) *On a compact conformally flat Riemannian manifold* (M^m, g) *with finite fundamental group, we have a conformally related Riemannian metric* g^* *whose scalar curvature is constant.*

(iii) *For a finite group* $\{\varphi_0 = \text{identity}, \varphi_1, \cdots, \varphi_{k-1}\}$ *of conformal transformations acting on* $S^m(1)$ *without fixed point, there are a conformal transformation* θ *and isometries* ϕ_i *on* $S^m(1)$ *such that* $\varphi_i = \theta^{-1}\cdot\phi_i\cdot\theta$, $i = 1, \cdots, k-1$.

With respect to the integral of the scalar curvature, we have

Theorem D. *Let* (M^m, g) *be a compact conformally flat Riemannian manifold with finite fundamental group. Then*

$$\int_M S dM > 0 ,$$

where dM *denotes the volume element of* $(M, {}^m g)$.

2. Proof of theorems

To prove our theorems we need to apply the following lemmas.

Lemma 1 (*Kuiper* [7]). *A conformally flat simply connected Riemannian manifold* (M^m, g) *is conformorphic to an open submanifold of* $S^m(1)$. *In particular, if* M^m *is compact, then* (M^m, g)*is comformorphic to* $S^m(1)$.

Lemma 2 (*Obata* [12, *Theorem* 1]). *There is no conformal transformation between compact* (M^m, g) *of constant scalar curvature* $S \geq 0$ *and* $('M^m, 'g)$ *of constant scalar curvature* $'S \leq 0$ *except for* $S = 'S = 0$.

Lemma 3 (*Obata* [14, *Proposition* 6.1]). *Let* g_0 *be the Riemannian metric of* $S^m(1)$, *and* g^* *another Riemannian metric on* S^m *conformal to* g_0. *Then* g^* *has constant scalar curvature* $S^* = m(m-1)$ *if and only if* g^* *has constant curvature* 1.

Lemma 4 (*Nagano* [11]). *Let* (M^m, g), $m \geq 3$, *be a complete Riemannian manifold with parallel Ricci tensor. If* (M^m, g) *admits a conformal transformation* f, *then one of the three cases occurs*: (1) f *is an isometry*, (2) f *is homothetic, and* (M^m, g) *is isometric to the Euclidean space*, (3) (M^m, g) *is isometric to* $S^m(K)$.

Lemma 5 (*Trudinger* [14, *Theorem* 2, *Corollary* 1]). *Let* (M^m, g) *be a compact Riemannian manifold with nonpositive total scalar curvature, i.e.,*

$$\int_M S dM \leq 0 ,$$

and $m \geq 3$. *Then there is a conformally related Riemannian metric* g^* *whose scalar curvature is nonpositive and constant.*

Proof of Theorem A. Let $({}^{\#}M^m, {}^{\#}g)$ be the universal covering Riemannian manifold of (M^m, g). Since the fundamental group of M^m is finite, M^m is compact. Since by the hypothesis $({}^{\#}M^m, {}^{\#}g)$ has constant scalar curvature ${}^{\#}S = S$, by Lemma 1 we have a conformal diffeomorphism f of $({}^{\#}M^m, {}^{\#}g)$ to $S^m(1)$, and therefore by Lemma 2 the scalar curvature S cannot be nonpositive. If we consider a homothetic deformation $g \to [S/(m(m-1))]g$, it follows that $[S/(m(m-1))]g$ has constant scalar curvature $m(m-1)$. Now define a Riemannian metric g^* on S^m by $g^* = [S/(m(m-1))]f^{-1*}{}^{\#}g$. Then g^* is conformal to g_0 and $S^* = m(m-1)$. By Lemma 3, g^* is of constant curvature 1, so that ${}^{\#}g$ and hence g are of constant curvature $S/[m(m-1)]$.

Proof of Theorem B. By S we denote the scalar curvature of (M^m, g). Then (M^m, g) is of positive constant curvature $K = S/[m(m-1)]$ by Theorem A, and is isometric to $S^m(K)$ by Lemma 4.

Proof of Theorem C. (i) $\Rightarrow$ (ii) is obvious.

We prove (ii) $\Rightarrow$ (iii). Let $G = \{\varphi_0, \varphi_1, \cdots, \varphi_{k-1}\}$ be a finite group of conformal transformations on $S^m(1)$ acting without fixed point. Since G is a compact group, we have a conformally related Riemannian metric g' such that G is isometric with respect to g' (cf. Ishihara [6], more precisely, $(g')_x = [g_x + (\varphi_1^* g)_x + \cdots + (\varphi_{k-1}^*)_x]/k, x \in S^m$). (S^m, g') is factorized by G and we get a Riemannian manifold (M^m, g'). Denote the projection by π. Because (M^m, g') is compact conformally flat with finite fundamental group, by (ii) we have a conformally related Riemannian metric g^* whose scalar curvature S^* is constant, so that we can assume that $S^* = m(m-1)$. Denoting $\pi^* g^*$ by g^* again, (S^m, g^*) is the universal covering Riemannian manifold of (M^m, g^*). Since φ_i induces an isometry φ_i^* of (S^m, g^*), $\theta' : x \in (S^m, g') \to x \in (S^m, g^*)$ is a conformal transformation, and therefore $\theta'' : (S^m, g_0) \to (S^m, g') \to (S^m, g^*)$ is also so. Thus we have $\varphi_i = \theta''^{-1} \cdot \varphi_i^* \cdot \theta''$, $i = 1, \cdots, k-1$. By Lemma 3, (S^m, g^*) is of constant curvature 1. Hence we obtain an isometry $\gamma : (S^m, g^*) \to (S^m, g_0)$. Putting $\theta = \gamma \cdot \theta''$ and $\phi_i = \gamma \cdot \varphi_i^* \cdot \gamma^{-1}$, we have $\varphi_i = \theta^{-1} \cdot \phi_i \cdot \theta$.

Finally we prove (iii) $\Rightarrow$ (i). Let (M^m, g) be a compact conformally flat Riemannian manifold with finite fundamental group G, and $({}^{\#}M^m, {}^{\#}g)$ its universal covering Riemannian manifold. Then we have a conformal transformation f' of $({}^{\#}M^m, {}^{\#}g)$ to $S^m(1)$. Let $\xi_i \in G$ be a covering transformation, and put $\varphi_i' = f' \cdot \xi_i \cdot f'^{-1}$. Then φ_i' is conformal with respect to g_0 on $S^m(1)$. By (iii) we have a conformal transformation θ and isometries ϕ_i on $S^m(1)$ such that $\varphi_i' = \theta^{-1} \cdot \phi_i \cdot \theta$ for $i = 1, \cdots, k-1$. Thus by defining φ_i by $f \cdot \xi_i \cdot f^{-1}$ for $f = \theta \cdot f'$, we obtain $\varphi_i = f \cdot \xi_i \cdot f^{-1} = \theta \cdot \varphi_i' \cdot \theta^{-1} = \phi_i =$ isometry. Now we show that we can define a conformally related Riemannian metric on M^m from a conformally related metric $f^* g_0$ on ${}^{\#}M^m$. Clearly, $f^* g_0$ is of constant curvature 1. Since $\xi_i = f^{-1} \cdot \phi_i \cdot f$, we have $\xi_i^*(f^* g_0) = f^* \cdot \phi_i^* \cdot f^{-1*} \cdot f^* g_0 = f^* \cdot \phi_i^* g_0 = f^* g_0$. Therefore G is isometric with respect to $f^* g_0$, and hence $f^* g_0$ is projectable on M^m.

Remark. So-called Yamabe's theorem [17] is in doubt (cf. Aubin [2], Eliasson [4], Trudinger [16]). If it is true, it gives some important property (iii) by Theorem C.

Proof of Theorem D. If the theorem is false, then the total scalar curvature is nonpositive, and therefore by Lemma 5 we have a conformally related Riemannian metric g^* on (M^m, g) such that the scalar curvature S^* is nonpositive and constant. However, this is impossible by Theorem A. Hence the theorem.

Remark. In Theorem D, "with finite fundamental group" is essential. In fact, if N^2 is a compact 2-dimensional Riemannian manifold (with genus ≥ 2) of negative constant curvature, and M^3 is a Riemannian product manifold $N^2 \times S^1$, where S^1 is a circle, then M^3 is compact conformally flat and has negative total scalar curvature.

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